

Multiparameter Heisenberg limit

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Using a quantum version of the Bell-Ziv-Zakai bound, I derive a Heisenberg limit to multiparameter estimation for any Gaussian prior probability density. The mean-square error lower bound is shown to have a universal quadratic scaling with respect to a quantum resource, such as the average photon number in the case of optical phase estimation, suitably weighted by the prior covariance matrix.

I. INTRODUCTION

The probabilistic nature of quantum mechanics imposes fundamental limits to information processing applications [1–4]. Such quantum limits have practical implications to many metrological applications, such as optical interferometry, optomechanical sensing, gravitational-wave detection [5–8], optical imaging [9–11], magnetometry, gyroscopy, and atomic clocks [12]. The existence of the so-called Heisenberg (H) limit to parameter estimation has in particular attracted much attention in recent years, as it implies that a minimum amount of resource, such as the average photon number for optical phase estimation, is needed to achieve a desired precision. After much debate and confusion [12–22], it has now been proven that the H limit indeed exists for the mean-square error of single-parameter estimation [23–27]. Although decoherence can impose stricter limitations [8, 28–34] than the H limit, the latter can still be relevant when the decoherence is relatively weak.

For many applications, such as waveform estimation [6, 8, 35] and optical imaging [36], the estimation of multiple parameters from measurements is needed [37–39]. In that case, the existence of a general H limit remains an open question. A recent work by Zhang and Fan [40] studies the quantum Ziv-Zakai bound (QZZB) [23] for multiple parameters, but they assume that the parameters are *a priori* independent, such that the single-parameter bound is applicable to each. In practice, and especially for the waveform estimation problem, the parameters often have nontrivial prior correlations, in which case a proper definition of the relevant quantum resource is unknown and the H limit remains to be proven.

Here I prove a multiparameter version of the H limit for any Gaussian prior. The proof uses the Bell-Ziv-Zakai bound (BZZB) [41, 42], which is an extension of the Ziv-Zakai family of bounds for single-parameter estimation [41]. The H limit is found to obey a universal quadratic scaling with respect to a quantum resource

suitably weighted by the prior covariance matrix. To illustrate the result, the bound is applied to the problem of optical phase waveform estimation, showing that an H limit can be defined with respect to the average photon number within the prior correlation time scale of the waveform.

II. QUANTUM BELL-ZIV-ZAKAI BOUND

Let x be a column vector of the unknown parameters, $P(x)$ be its prior probability density, $P(y|x)$ be the likelihood function with observation y , and $\tilde{x}(y)$ be the estimator. The mean-square error covariance matrix is defined as [43]

$$\Sigma \equiv \int dx dy P(y|x) P(x) [\tilde{x}(y) - x] [\tilde{x}(y) - x]^\top, \quad (2.1)$$

where $^\top$ denotes the transpose. One useful version of the BZZB is given by [41, 42]

$$u^\top \Sigma u \geq \int_0^\infty d\tau \tau \max_{v: u^\top v = 1} \int dx \min [P(x), P(x + v\tau)] \times P_e(x, x + v\tau), \quad (2.2)$$

where u is an arbitrary real vector and $P_e(x^{(0)}, x^{(1)})$ is the error probability in discriminating equally likely hypotheses $x = x^{(0)}$ and $x = x^{(1)}$ from an observation y with the likelihood function $P(y|x)$. If $P_e(x, x + v\tau)$ does not depend on x , the x integral in Eq. (2.2) depends only on the prior distribution $P(x)$. For a Gaussian $P(x)$ with covariance matrix Σ_0 [42],

$$\begin{aligned} \int dx \min [P(x), P(x + v\tau)] &= \operatorname{erfc} \frac{\tau}{\tau_0}, \\ \operatorname{erfc} z &\equiv \frac{2}{\sqrt{\pi}} \int_0^z d\xi \exp(-\xi^2), \\ \tau_0 &\equiv \left(\frac{8}{v^\top \Sigma_0^{-1} v} \right)^{1/2}. \end{aligned} \quad (2.3)$$

The erfc function is plotted in Fig. 1.

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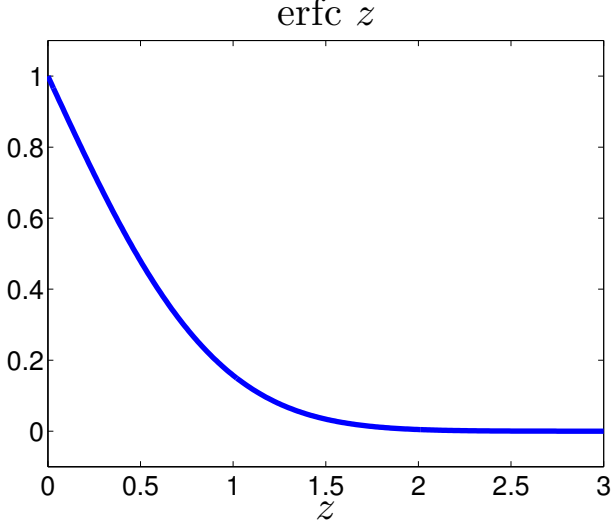


FIG. 1. The erfc function.

Suppose now that a quantum probe is used to measure the parameters. The likelihood function becomes

$$P(y|x) = \text{tr } E(y)\rho_x, \quad (2.4)$$

where $E(y)$ is the positive operator-valued measure (POVM) that describes the measurement and ρ_x is the density operator conditioned on the unknown x . The following quantum bound can be used [44]:

$$P_e(x, x + v\tau) \geq \frac{1}{2} \left[1 - \sqrt{1 - F(\rho_x, \rho_{x+v\tau})} \right], \quad (2.5)$$

where

$$F(\rho_x, \rho_{x+v\tau}) \equiv \left(\text{tr } \sqrt{\sqrt{\rho_x} \rho_{x+v\tau} \sqrt{\rho_x}} \right)^2 \quad (2.6)$$

is the Uhlmann fidelity between ρ_x and $\rho_{x+v\tau}$. This quantum bound, together with the BZZB, results in a quantum Bell-Ziv-Zakai bound (QBZZB) on the mean-square error of multiparameter estimation, just like the single-parameter case [23]. It is possible to define QBZZBs for error functions other than the mean-square criterion [41, 42], although I shall focus on the mean-square error here because of its popularity.

III. QUANTUM PHASE ESTIMATION

Suppose that the density operator is

$$\rho_x = U_x \rho U_x^\dagger, \quad (3.1)$$

and the unitary has the following form:

$$U_x = \exp(ix^\top n) = \exp\left(i \sum_j x_j n_j\right), \quad (3.2)$$

where n is a column vector of quantum operators and ρ is the initial density operator. Assuming that $|\psi\rangle$ is a purification of ρ and defining

$$\langle O \rangle \equiv \langle \psi | O | \psi \rangle, \quad (3.3)$$

a lower bound on the fidelity is given by

$$F(\rho_x, \rho_{x+v\tau}) \geq |\langle \exp(i\tau v^\top n) \rangle|^2 \quad (3.4)$$

$$= \sum_{m,l} P_m P_l \exp[i\tau v^\top (m-l)] \quad (3.5)$$

$$= \sum_{m,l} P_m P_l \cos[\tau v^\top (m-l)], \quad (3.6)$$

where

$$P_m \equiv |\langle m | \psi \rangle|^2 \quad (3.7)$$

is the probability distribution with respect to the n eigenstates.

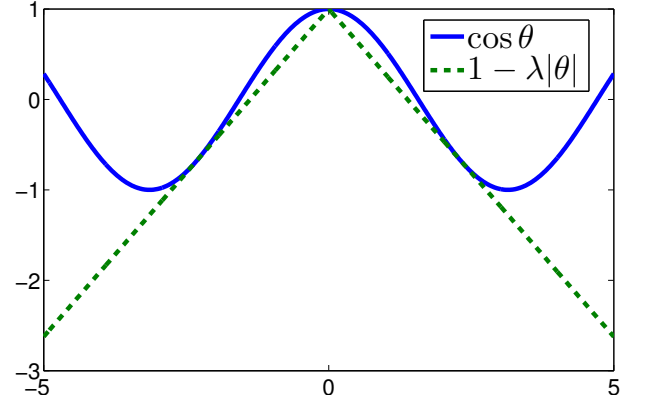


FIG. 2. A lower bound for cosine.

A useful bound for the cosine function for deriving the H limit is [23]

$$\cos \theta \geq 1 - \lambda |\theta|, \quad (3.8)$$

where $\lambda \approx 0.7246$ is a solution of $\lambda = \sin \phi = (1 - \cos \phi)/\phi$, as shown in Fig. 2. Substituting this bound into Eq. (3.6) and using the triangle inequality, one obtains

$$F \geq \sum_{m,l} P_m P_l [1 - \lambda \tau |v^\top (m-l)|] \quad (3.9)$$

$$\geq \sum_{m,l} P_m P_l [1 - \lambda \tau (|v^\top m - H_0| + |v^\top l - H_0|)] \quad (3.10)$$

$$= 1 - 2\lambda \tau \langle |v^\top n - H_0| \rangle, \quad (3.11)$$

where H_0 is an arbitrary constant. It is possible to obtain a slightly tighter bound numerically using the method

in Refs. [27, 45], but Eq. (3.11) will produce the same scaling. Since $0 \leq F \leq 1$, a tighter lower bound is

$$F \geq \Lambda\left(\frac{\tau}{\tau_F}\right) \equiv \begin{cases} 1 - \tau/\tau_F, & \tau < \tau_F, \\ 0, & \tau \geq \tau_F, \end{cases} \quad (3.12)$$

$$\tau_F \equiv \frac{1}{2\lambda\langle |v^\top n - H_0| \rangle},$$

as shown in Fig. 3.

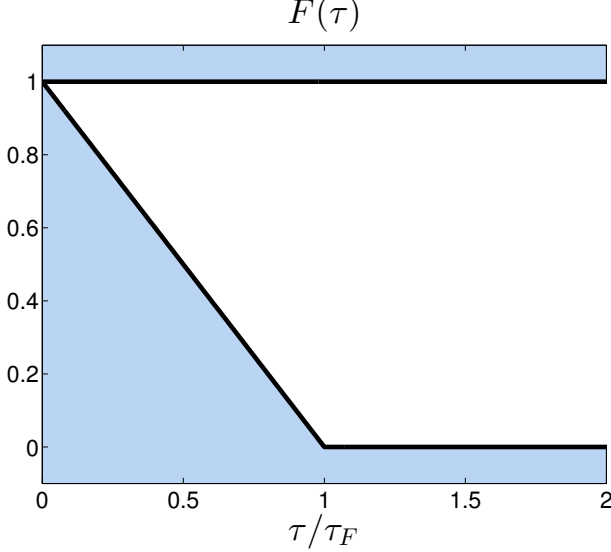


FIG. 3. Bounds for the fidelity. The white area is the permissible area.

Putting Eqs. (2.3), (2.5), and (3.12) together,

$$\begin{aligned} & \max_{v: u^\top v=1} \int dx \min[P(x), P(x+v\tau)] P_e(x, x+v\tau) \\ & \geq \frac{1}{2} \max_{v: u^\top v=1} \operatorname{erfc}\left(\frac{\tau}{\tau_0}\right) \Lambda\left(\sqrt{\frac{\tau}{\tau_F}}\right). \end{aligned} \quad (3.13)$$

Recall that τ_0 and τ_F depend on v . The maximization does not seem to be tractable analytically, so I choose a v that maximizes only the erfc function:

$$v_0 \equiv \arg \max_{v: u^\top v=1} \operatorname{erfc}\left(\frac{\tau}{\tau_0}\right) = \frac{\Sigma_0 u}{u^\top \Sigma_0 u}, \quad (3.14)$$

such that

$$\begin{aligned} & \max_{v: u^\top v=1} \operatorname{erfc}\left(\frac{\tau}{\tau_0}\right) \Lambda\left(\sqrt{\frac{\tau}{\tau_F}}\right) \\ & \geq \operatorname{erfc}\left(\frac{\tau}{\tau_0}\right) \Lambda\left(\sqrt{\frac{\tau}{\tau_F}}\right) \Big|_{v=v_0}, \\ & \tau_0(v_0) = 2\sqrt{2u^\top \Sigma_0 u}, \\ & \tau_F(v_0) = \frac{1}{2\lambda\langle |u^\top \Sigma_0 n / (u^\top \Sigma_0 u) - H_0| \rangle}. \end{aligned} \quad (3.15)$$

Combining Eqs. (2.2), (3.13), and (3.15) then produces the following bound:

$$u^\top \Sigma u \geq Z \equiv \frac{1}{2} \int_0^{\tau_F} d\tau \operatorname{erfc}\left(\frac{\tau}{\tau_0}\right) \left(1 - \sqrt{\frac{\tau}{\tau_F}}\right) \Big|_{v=v_0} \quad (3.16)$$

The integral can be computed numerically, as shown in Fig. 4, but there are two analytic limits of interest:

1. The prior-information limit ($\tau_F \gg \tau_0$):

$$\lim_{\tau_F/\tau_0 \rightarrow \infty} Z = \frac{\tau_0^2}{8} = u^\top \Sigma_0 u, \quad (3.17)$$

where the bound is determined only by the prior covariance matrix, as expected;

2. The asymptotic limit ($\tau_F \ll \tau_0$), where the measurement provides much more information:

$$\begin{aligned} \lim_{\tau_0/\tau_F \rightarrow \infty} Z &= \frac{\tau_F^2}{20} = \frac{1}{80\lambda^2 H_+^2}, \\ H_+ &\equiv \left\langle \left| \frac{u^\top \Sigma_0 n}{u^\top \Sigma_0 u} - H_0 \right| \right\rangle, \end{aligned} \quad (3.18)$$

and H_+ quantifies the relevant resource for the estimation. Eq. (3.18) is the central result of this paper and an appropriate generalization of the single-parameter case [23].

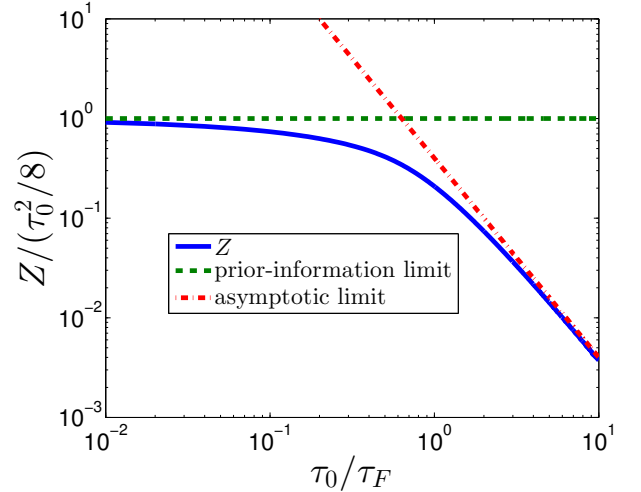


FIG. 4. A quantum lower error bound Z on $u^\top \Sigma u$ versus the parameter τ_0/τ_F in log-log scale, the prior-information limit $Z \rightarrow \tau_0^2/8$, and the asymptotic H limit $Z \rightarrow \tau_F^2/20$.

For example, the error bound for estimating a particular parameter x_k can be obtained by setting u as

$$u_j = \delta_{jk}, \quad (3.19)$$

$$u^\top \Sigma u = \Sigma_{kk} \geq Z_k \rightarrow \frac{1}{80\lambda^2 H_{+k}^2}, \quad (3.20)$$

$$H_{+k} \equiv \left\langle \left| \frac{1}{\Sigma_{0kk}} \sum_l \Sigma_{0kl} n_l - H_0 \right| \right\rangle. \quad (3.21)$$

For optical phase estimation with n_l being a photon number operator, one can assume $H_0 = 0$ and use the triangle inequality to obtain

$$H_{+k} \leq \frac{1}{\Sigma_{0kk}} \sum_l |\Sigma_{0kl}| \langle n_l \rangle, \quad (3.22)$$

which produces an H limit with respect to a weighted average of the photon numbers. The weighting of the photon numbers with respect to the prior covariance matrix is the key feature of the bound, as it properly accounts for the optical modes that can contribute to the estimation of a particular phase.

A special case is when the parameters are independent *a priori*, such that

$$\Sigma_{0kl} = \Sigma_{0kk} \delta_{kl}, \quad (3.23)$$

$$H_{+k} = \langle |n_k - H_0| \rangle, \quad (3.24)$$

and the single-parameter bound [23] is recovered. Zhang and Fan used this [40] to rule out any significant quantum enhancement with a proposal by Humphreys *et al.* for quantum multiparameter estimation [36].

IV. OPTICAL PHASE WAVEFORM ESTIMATION

To illustrate the result derived in the previous section, consider the continuous-time limit of the QBZZB for optical phase estimation. The photon number of each mode is related to the photon flux $I(t)$ and the time duration dt of the mode:

$$n_l = dt I(t_l). \quad (4.1)$$

The mean-square error for each phase parameter becomes the error for estimating the phase at a certain time:

$$\Sigma_{kk} = \Sigma(t_k, t_k), \quad (4.2)$$

and the H limit becomes

$$\Sigma(t, t) \geq Z(t) \rightarrow \frac{1}{80\lambda^2 H_+^2(t)}, \quad (4.3)$$

$$H_+(t) \equiv \left\langle \left| \frac{1}{\Sigma_0(t, t)} \int dt' \Sigma_0(t, t') I(t') - H_0 \right| \right\rangle \quad (4.4)$$

$$\leq \frac{1}{\Sigma_0(t, t)} \int dt' |\Sigma_0(t, t')| \langle I(t') \rangle. \quad (4.5)$$

The relevant resource H_+ is defined as the time integral of the average photon flux $\langle I(t') \rangle$ weighted by the prior covariance function $\Sigma_0(t, t')$. For example, for the Ornstein-Uhlenbeck process,

$$\Sigma_0(t, t') = \sigma_0 \exp\left(-\frac{|t - t'|}{T_0}\right), \quad (4.6)$$

$$H_+(t) \leq \int_{-\infty}^{\infty} dt' \exp\left(-\frac{|t - t'|}{T_0}\right) \langle I(t') \rangle, \quad (4.7)$$

which states that only the optical modes within the prior time scale T_0 can contribute to the estimation at a particular time.

If $\langle I \rangle$ is constant in time, $H_+(t) \propto \langle I \rangle$, and there exists a universal quadratic error scaling $\propto 1/\langle I \rangle^2$ for any Gaussian prior. Tighter scalings can be derived for Gaussian quantum states [35], but the H limit is still valuable as a simple and more general no-go theorem.

V. CONCLUSION

To conclude, I have proved an H limit with a universal $1/N^2$ scaling for multiparameter estimation with any Gaussian prior, where N is an appropriately defined quantum resource. The key feature of the bound is the use of the prior covariance matrix to define N , enabling a proper accounting of the relevant quantum resources. In the case of optical phase waveform estimation, the H limit implies the intuitive result that only the optical modes within the prior correlation time scale can contribute to the estimation at a particular time.

It should be emphasized that the H limit derived here may well not be attainable and the quantum Cramér-Rao bound [6, 8, 35] may provide tighter bounds for more specific quantum states, but the generality and simplicity of the result here should still be valuable as a no-go theorem. It may also be possible to derive tighter bounds or study other priors using the present formalism. These possibilities are left for future investigations.

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- [1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
 [2] V. Giovannetti, S. Lloyd, and L. Maccone, *Science* **306**, 1330 (2004).

- [3] V. Giovannetti, S. Lloyd, and L. Maccone, *Nature Photon.* **5**, 222 (2011).
 [4] A. S. Holevo, *Quantum Systems, Channels, Information* (de Gruyter, Berlin, 2012).

- [5] V. B. Braginsky and F. Y. Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
- [6] M. Tsang, H. M. Wiseman, and C. M. Caves, *Phys. Rev. Lett.* **106**, 090401 (2011).
- [7] M. Tsang and R. Nair, *Phys. Rev. A* **86**, 042115 (2012).
- [8] M. Tsang, *New Journal of Physics* **15**, 073005 (2013).
- [9] N. Treps, N. Grosse, W. P. Bowen, C. Fabre, H.-A. Bachor, and P. K. Lam, *Science* **301**, 940 (2003), <http://www.sciencemag.org/content/301/5635/940.full.pdf>.
- [10] M. Tsang, *Phys. Rev. Lett.* **102**, 253601 (2009).
- [11] M. A. Taylor, J. Janousek, V. Daria, J. Knittel, B. Hage, H.-A. Bachor, and W. P. Bowen, *Nature Photonics* **7**, 229 (2013).
- [12] J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, *Phys. Rev. A* **54**, R4649 (1996).
- [13] B. Yurke, S. L. McCall, and J. R. Klauder, *Phys. Rev. A* **33**, 4033 (1986).
- [14] B. C. Sanders and G. J. Milburn, *Phys. Rev. Lett.* **75**, 2944 (1995).
- [15] Z. Y. Ou, *Phys. Rev. Lett.* **77**, 2352 (1996).
- [16] M. Zwiernik, C. A. Pérez-Delgado, and P. Kok, *Phys. Rev. Lett.* **105**, 180402 (2010).
- [17] M. Zwiernik, C. A. Pérez-Delgado, and P. Kok, *Phys. Rev. Lett.* **107**, 059904 (2011).
- [18] Á. Rivas and A. Luis, *New Journal of Physics* **14**, 093052 (2012).
- [19] A. Luis and A. Rodil, *Phys. Rev. A* **87**, 034101 (2013).
- [20] A. Luis, *Annals of Physics* **331**, 1 (2013).
- [21] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. D. Huver, H. Lee, and J. P. Dowling, *Phys. Rev. Lett.* **104**, 103602 (2010).
- [22] Y. R. Zhang, G. R. Jin, J. P. Cao, W. M. Liu, and H. Fan, *Journal of Physics A: Mathematical and Theoretical* **46**, 035302 (2013).
- [23] M. Tsang, *Phys. Rev. Lett.* **108**, 230401 (2012).
- [24] V. Giovannetti, S. Lloyd, and L. Maccone, *Phys. Rev. Lett.* **108**, 260405 (2012).
- [25] M. J. W. Hall, D. W. Berry, M. Zwiernik, and H. M. Wiseman, *Phys. Rev. A* **85**, 041802 (2012).
- [26] R. Nair, ArXiv e-prints (2012), [arXiv:1204.3761](https://arxiv.org/abs/1204.3761) [quant-ph].
- [27] V. Giovannetti and L. Maccone, *Phys. Rev. Lett.* **108**, 210404 (2012).
- [28] S. Knysh, V. N. Smelyanskiy, and G. A. Durkin, *Phys. Rev. A* **83**, 021804 (2011).
- [29] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, *Nature Physics* **7**, 406 (2011).
- [30] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, *Brazilian Journal of Physics* **41**, 229 (2011).
- [31] B. M. Escher, L. Davidovich, N. Zagury, and R. L. de Matos Filho, *Phys. Rev. Lett.* **109**, 190404 (2012).
- [32] C. L. Latune, B. M. Escher, R. L. de Matos Filho, and L. Davidovich, ArXiv e-prints (2012), [arXiv:1210.3316](https://arxiv.org/abs/1210.3316) [quant-ph].
- [33] R. Demkowicz-Dobrzański, J. Kołodyński, and M. Guţă, *Nature Communications* **3**, 1063 (2012), [arXiv:1201.3940](https://arxiv.org/abs/1201.3940) [quant-ph].
- [34] S. I. Knysh, E. H. Chen, and G. A. Durkin, ArXiv e-prints (2014), [arXiv:1402.0495](https://arxiv.org/abs/1402.0495) [quant-ph].
- [35] D. W. Berry, M. J. W. Hall, and H. M. Wiseman, *Phys. Rev. Lett.* **111**, 113601 (2013).
- [36] P. C. Humphreys, M. Barbieri, A. Datta, and I. A. Walmsley, *Phys. Rev. Lett.* **111**, 070403 (2013).
- [37] H. P. Yuen and M. Lax, *Information Theory, IEEE Transactions on* **19**, 740 (1973).
- [38] C. Helstrom and R. Kennedy, *Information Theory, IEEE Transactions on* **20**, 16 (1974).
- [39] M. G. Genoni, M. G. A. Paris, G. Adesso, H. Nha, P. L. Knight, and M. S. Kim, *Phys. Rev. A* **87**, 012107 (2013).
- [40] Y.-R. Zhang and H. Fan, ArXiv e-prints (2014), [arXiv:1402.6197](https://arxiv.org/abs/1402.6197) [quant-ph].
- [41] H. L. Van Trees and K. L. Bell, eds., *Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking* (Wiley-IEEE, Piscataway, 2007), and references therein.
- [42] K. Bell, Y. Steinberg, Y. Ephraim, and H. Van Trees, *IEEE Transactions on Information Theory* **43**, 624 (1997).
- [43] H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part I*. (John Wiley & Sons, New York, 2001).
- [44] C. A. Fuchs and J. van de Graaf, *IEEE Trans. Inf. Theor.* **45**, 1216 (1999).
- [45] V. Giovannetti, S. Lloyd, and L. Maccone, *Phys. Rev. A* **67**, 052109 (2003).